Thm 9.2.18. Assume that the SCG $\mathcal{G}$ Satisfies (CG3') (CG4').
Then $\forall X \in \mathcal{X}$, $\Delta^{Xre} = \Delta_+^{Xre} \cup -\Delta_+^{Xre}$ and
$$\Delta_+^{Xre} = \underline{\bigcup_{k} \Lambda^X(k)}$$
where the union is taken over all X-reduced sequence k.

proof. Let $\underline{\alpha \in \Delta^{Xre}}$, $l \geq 0$, $k = (i_1, \cdots, i_l) \in I^l$ $i \in I$
s.t. $\underline{\alpha = \mathrm{id}_X s_{i_1} \cdots s_{i_l}(\alpha_i)}$. Let $k' = (i_1, \cdots, i_l, i)$
Assume $\underline{\alpha \neq \mathrm{id}_X s_{j_1} \cdots s_{j_k}(\alpha_j)}$ $\forall$ $0 \leq \underline{k} < l$, $j \in I$ $(j_1 \cdots j_k) \in I^k$

○ Assume $\underline{k \text{ is } X\text{-reduced}}$ $\underline{\alpha \in \mathbb{N}_0^I}$. Then
$\underline{\Lambda^X(k') \subset \mathbb{N}_0^I}$ by assumption. $\alpha_{\Uparrow} = \beta_{l+1}^{X,k'} \in \mathbb{N}_0^I$
$\Lambda^X(k') =$

k' is X-reduced by Lem 9.2.5
Assume $\underline{k \text{ is } X\text{-reduced}}$ $\underline{\alpha \notin \mathbb{N}_0^I}$. Then by prop 9.2.16
$\exists$ X-reduced seq. $(j_1, \cdots, j_l) \in I^l$ s.t $j_l = i$ and
$\mathrm{id}_X s_{i_1} \cdots s_{i_l} = \underline{\mathrm{id}_X s_{j_1} \cdots s_{j_l}} \longrightarrow k''$
$/\!/\ s_i$

$\underline{\alpha = \mathrm{id}_X s_{i_1} \cdots s_{i_l}(\alpha_i)} = \mathrm{id}_X s_{j_1} \cdots \underline{s_{j_l}}(\alpha_i)$
$= -\underline{\mathrm{id}_X s_{j_1} \cdots s_{j_{l-1}}(\alpha_i)}$ $\quad \underline{-\alpha \in \Delta_+^{Xre}}$ $\Uparrow$
$= -\beta_l^{X,k''} \in \underline{-\Lambda^X(j_1, \cdots, j_l)}$

Finally, assume $\underline{k \text{ is not } X\text{-reduced}}$. Then by Lem 9.2.5
and by assumption, $\exists$ $2 \leq h \leq l$ s.t $\beta_j^{X,k} \in \mathbb{N}_0^I$ $1 \leq j < h$
$\forall$ $\underline{\beta_h^{X,k} \notin \mathbb{N}_0^I}$ $\left(\beta_1 = \alpha_{i_1} \in \mathbb{N}_0^I\right)$
$\quad /\!/\ \mathrm{id}_X s_{i_1} \cdots s_{i_{h-1}}(\alpha_{i_h})$
Hence by prop. 9.2.16, $\exists$ $j_1, \cdots, j_{h-1} \in I$ s.t. $j_{h-1} = i_h$
and $\mathrm{id}_X s_{i_1} \cdots s_{i_{h-1}} = \mathrm{id}_X s_{j_1} \cdots \underline{s_{j_{h-1}}}$
$\quad /\!/\ s_{i_h}$

$\alpha = \mathrm{id}_X\, s_{i_1} \cdots s_{i_{h-1}}\, s_{i_h} \cdots s_{i_l}\,(\alpha_i)$

$= \mathrm{id}_X\, s_{j_1} \cdots \underbrace{s_{j_{h-1}}\, s_{i_h}}_{=1} \cdots s_{i_l}\,(\alpha_i)$

a contradiction. #

Lem 9.2.19. Let J be a finite set, $i, j \in J$, $w \in \mathrm{Aut}(\mathbb{Z}^J)$

assume $w(\alpha_k) \in \alpha_k + \mathbb{Z}\alpha_i + \mathbb{Z}\alpha_j \quad \forall k \in J$

$w(\alpha_j), w^{-1}(\alpha_j) \in \mathbb{N}_0^J \cup -\mathbb{N}_0^J$. If $\det(w) = 1$, $w(\alpha_i) = \alpha_i$

then $w(\alpha_j) = \alpha_j$

If additionally $w(\alpha_k), w^{-1}(\alpha_k) \in \mathbb{N}_0^J \cup -\mathbb{N}_0^J \quad \forall k \in J$

then $w = \mathrm{id}$.

proof If $i = j$, the first claim holds.

Assume $i \neq j$. $w(\alpha_j) = a\alpha_i + b\alpha_j$ for some $a, b \in \mathbb{Z}$.

Then $b = 1$ since $\det(w) = 1$. $w(\alpha_i) = \alpha_i$, $w(\alpha_k) \in \alpha_k + \mathbb{Z}\alpha_i + \mathbb{Z}\alpha_j$

$$\left[\; w(\alpha_i, \alpha_j, \alpha_k \cdots) = (\alpha_i, \alpha_j, \alpha_k, \cdots) \begin{bmatrix} 1 & a & & * \\ 0 & b & & \\ \vdots & \vdots & 1 & \\ 0 & 0 & & \ddots \; 1 \end{bmatrix} \qquad \det(w) = b \right.$$

we conclude $w^{-1}(\alpha_j) = -a\alpha_i + \alpha_j$

Therefore $a = 0$ since $w(\alpha_j), w^{-1}(\alpha_j) \in \mathbb{N}_0^J \cup -\mathbb{N}_0^J$.

Hence $w(\alpha_j) = \alpha_j$.

The second claim is similar.

$w(\alpha_k) = \alpha_k + a\alpha_i + b\alpha_j \qquad k \in J \setminus \{i, j\}$

$w^{-1}(\alpha_k) = \alpha_k - a\alpha_i - b\alpha_j$

$a = b = 0$ since $w(\alpha_k), w^{-1}(\alpha_k) \in \mathbb{N}_0^J \cup -\mathbb{N}_0^J$. #

Cor. 9.2.20. For any SCG TFAE:

(1) $\mathcal{G}$ satisfies (CG3'), (CG4')

(2) $\mathcal{G}$ is a CG.

Moreover, if $\mathcal{G}$ satisfies (CG3) then $\underline{m^x_{ij} = \bar{m}^x_{ij}}$ $\forall x$, $i \neq j$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ $\parallel$

proof. Suppose $\underline{(CG3)}$. $\quad |\Delta^{x\mathrm{re}} \cap \mathbb{N}_0\alpha_i + \mathbb{N}_0\alpha_j)|$

Then (CG3') holds because of Lem 9.2.7.

We prove $m^x_{ij} = \bar{m}^x_{ij}$

Let $x \in \mathcal{X}$, $i, j \in I$, $i \neq j$. Then $\underline{m^x_{ij}} \geq \bar{m}^x_{ij}$ because of Lem 9.2.7 (1).

[$K = (i, j, \cdots)$ 長さ $\bar{m}^x_{ij}$ $\quad$ K is x-reduced.

$\Lambda^x(K) = \{\beta_1, \ldots, \beta_l\} \subseteq \mathbb{N}_0^I$ $\quad$ by Lem 9.2.7 (1).

$\beta_1 = \alpha_i$, $\beta_2 = \mathrm{id}_x s_i(\alpha_j)$, $\cdots$, $\beta_l \in \mathbb{N}_0\alpha_j + \mathbb{N}_0\alpha_i$

$\underbrace{\Lambda^x(K)}_{l = \bar{m}^x_{ij}} \subset \underbrace{\Delta^{x\mathrm{re}}_+ \cap \mathbb{N}_0\alpha_i + \mathbb{N}_0\alpha_j}_{m^x_{ij}}$]

If $\bar{m}^x_{ij} = \infty \Rightarrow m^x_{ij} = \infty$

Assume $m = \bar{m}^x_{ij} < \infty$ $\quad$ $w = \mathrm{id}_x s_{i_1} \cdots s_{i_m}$

$(i_1, \ldots, i_m) = K^x_{ij}$

Since $\underline{K^x_{ij}}$ is x-reduced $\quad \Lambda^x(K^x_{ij}) \subset \mathbb{N}_0^I$ by (CG3').

Thus $\underline{w(\alpha_i),\ w(\alpha_j) \notin \mathbb{N}_0^I}$ by the defn of K^x_{ij} and by Lem 9.2.5.

$\underline{K' = (i, \ldots, i_m, \overset{j}{i})}$ $\quad$ $\underbrace{w(\alpha_i)}_{\in \mathbb{N}_0} = \Lambda^x(K')$

Hence $\underline{w(\alpha_i)}, w(\alpha_j) \in -\mathbb{N}_0^I$ by (CG3)

and therefore

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad m^x_{ij} \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \bar{m}^x_{ij}$

$\Delta^{x\mathrm{re}} \cap (\mathbb{N}_0\alpha_i + \mathbb{N}_0\alpha_j) \subset \Delta^{x\mathrm{re}}_+ \ldots \Lambda^x(K^x_{ij}) \ldots$

$\underline{\Delta \cap (\mathbb{N}_0\alpha_i + \mathbb{N}_0\alpha_j)} \subset \Delta \cap (w^{-1}(R_{ij}))$ by Lem 9.1.1(2)

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$\underline{a}\alpha_i + b\alpha_j \xrightarrow{w^{-1}} a\,w^{-1}(\alpha_i) + b\,\underline{w^{-1}(\alpha_j)} \;\underline{\in -\mathbb{N}_0}\,?$

We conclude $m^x_{ij} = \bar{m}^x_{ij}$.

$(2) \Rightarrow (1)$ Assume $\mathcal{G}$ is a CG. Then CG3' holds

$m^x_{ij} = \bar{m}^x_{ij} \quad \forall x \in X,\; i \neq j \in I$

Therefore (CG4') follows from (CG4) Lem $\underline{9.2.15(1)}$, Lem 9.2.19.

$$\begin{array}{cc} \Updownarrow & \Updownarrow \\ (CG4) + \underline{id_x(s_is_j)^m(\alpha_k) = \alpha_k} & (r_ir_j)^{m_{ij}}(x) = x \end{array}$$

with $w = id(s_is_j)^m$, $m = \bar{m}_{ij}$

$(1) \Rightarrow (2)$ Assume (1). Then (CG3) holds by Thm 9.2.18.

Then $m^x_{ij} = \bar{m}^x_{ij}$.

(CG4) hold because of (CG4'). #

Def. 9.2.21. $\forall$ SG $\mathcal{G}$, $\mathcal{G} = \mathcal{G}(I, X, r, A)$ $\forall x \in X$, $i, j \in I$

$\text{prod}^x_{ij}(2k) = id_x(s_is_j)^k$

$\text{prod}^x_{ij}(2k+1) = id_x(s_is_j)^k s_i \quad \in \text{Hom}(W(\mathcal{G}), X)$

Cor. 9.2.22. Let $\mathcal{G} = \mathcal{G}(I, X, r, A)$ be a CG.

Let $x \in X$, $i, j \in I$, $i \neq j$. If $m^x_{ij} < \infty$ then

(1) $id_x(s_is_j)^{m^x_{ij}} = id_x$

(2) $\text{prod}^x_{ij}(\underline{m^x_{ij}}) = \text{prod}^x_{ji}(m^x_{ij})$ (Coxeter relations)

proof. (1) follow from Cor. 9.2.20, Lem 9.2.15.

$\left[\; \text{Cor. } 9.2.20 \Rightarrow (CG3')(CG4') \xrightarrow{\text{Lem } 9.2.15} id(s_is_j)^m = id_x \;\right]$

(2) follows from (1).

[$m = 2k$, (1) $\Rightarrow$ $(s_i s_j)^m = id \Longrightarrow (s_i s_j)^{\frac{m}{2}} (s_i s_j)^{\frac{m}{2}} = id$

$$\Longrightarrow \underbrace{(s_i s_j)^{\frac{m}{2}}}_{\mathrm{prod}_{ij}(m_{ij}^{x})} = \underbrace{(s_j s_i)^{\frac{m}{2}}}_{\mathrm{prod}_{ji}(m_{ij}^{x})}$$]

Thm. 9.2.23. Let $G = G(I, X, r, A)$ be a SCG. Let $x \in X$, $i, j \in I$ $i \neq j$. Let $Y \subset X$, $x \in Y$ $r_i(Y) \cup r_j(Y) \subset Y$, and assume $\Delta^{Y\,re} \subset \mathbb{N}_0^I \cup -\mathbb{N}_0^I$ $\forall Y \in Y$.

If $|m_{ij}^x| < \infty$ then

$$m_{ij}^x = \min\{ n \geq 1 \mid F(id_x (s_i s_j)^n) = id_{\mathbb{Z}^I} \}$$

If $m_{ij}^x = \infty$, then $\forall n \geq 1$ $F(id_x(s_i s_j)^n) \neq id_{\mathbb{Z}^I}$.

proof. We assume G is connected. Then, since G satisfies (CG3). $m_{ij}^x = \bar{m}_{ij}^x$ by Cor. 9.2.20.

Moreover, (CG3') hold by Lem. 9.2.7 (1). $X \longrightarrow Y$

Assume $m_{ij}^x < \infty$. Then $F(id_x (s_i s_j)^{m_{ij}^x}) = id_{\mathbb{Z}^I}$ by Lem 9.2.15 and Lem 9.2.19.

[$w = id_x (s_i s_j)^{m_{ij}^x} \neq id$

$w(\alpha_k) \in \alpha_k + \mathbb{Z}\alpha_i + \mathbb{Z}\alpha_j$
$w(\alpha_j), w^{-1}(\alpha_j) \in \mathbb{N}_0^I \cup -\mathbb{N}_0^I$
$w(\alpha_i) = \alpha_i$
$w(\alpha_k), w^{-1}(\alpha_k) \in \mathbb{N}_0^I \cup -\mathbb{N}_0^I$

Lem 9.2.15 $\quad id_X(s_i s_j)^m(\alpha_i) = \alpha_i$

Now it suffices to prove $\underline{\underline{N(id_X(s_i s_j)^n) > 0}}$

$1 \le n < m_{ij}^X$.

$N(prod_{ij}^Y(\overline{m}_{ij}^X)) = m_{ij}^X$

[$\overline{m}_{ij}^X = 2k, \quad w = id_X(s_i s_j)^k, \quad k_i = (i, j, \dots)$

$N(prod_{ij}^Y(\overline{m}_{ij}^X)) = |\Delta^{Xre}(w)| = \underline{|\Lambda^X(k_i)|} = 2k = \overline{m}_{ij}^X = m_{ij}^X$]

$(s_i s_j)^n$ 可通过 $prod_{ij}(m_{ij})$ 得到

1. $m = 2k$

$\underline{n < m_{ij}^X}$

$N(id_X(s_i s_j)^{\frac{m}{2}+1}) \ge m-2 > 0$

$\vdots$

$N(id(s_i s_j)) \ge 2$

If $m_{ij}^X = \infty \Rightarrow k_{ij}^X$ has infinite length hence

$id_X(s_i s_j)^n(\alpha_i) \neq \alpha_i \quad \forall\ n \ge 1$ by Lem 9.2.7.

[否则会出现重复的 β_k]

#

Example 9.2.24. SCG Satisfying (CG3'), (CG4')(1)

not (CG4')(2).

$I = \{1, 2, 3\}, \quad X = \{1, 2, 3, 4\}, \quad r_1 = (12)(34)$

$r_2 = (23), \quad r_3 = id_X$

$A_1 = A_4 = \begin{bmatrix} 2 & -2 & -2 \\ -2 & 2 & -2 \\ -2 & -2 & 2 \end{bmatrix}, \quad A_m = \begin{bmatrix} 2 & -2 & -2 \\ -2 & 2 & 0 \\ -p_m & 0 & 2 \end{bmatrix}, \quad m \in \{2, 3\}$

$2 \le p_2 \lneq p_3$

Then $\mathcal{G}(I, X, r, A)$ is SCG.

$\Lambda = \Lambda_2 = \{(a, b, c) \mid a, b, c \in \mathbb{N}_0,\ a < b \neq c\}$

$P_1 = \{ (a,b,c) \mid \cdots \quad a > b + c \}$

$P_2 = \{ \cdots \mid \quad b > a + c$

$P_3 = \{ \cdots \quad c > a + b \}$

$\Lambda_1 = \Lambda_4 = P_2 \cup P_3$

Then $\bar{m}^X_{23} = \bar{m}^X_{32} = 2 \quad X \in \{2, 3\}$.

$k = (2, 3, 2, 3, \cdots)$

$\underline{\beta_1 = \alpha_{i_1} = \alpha_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \ \beta_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}, \quad \underline{\beta_3 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} = -\beta_1}$

(2) $\forall X \in \mathcal{X}$, $k = (i_1, \cdots, i_l) \in I^l$ $l \geq 1$, is not X-reduced if

(a) $\exists\, 1 \leq h < l$ s.t. $i_h = i_{h+1}$ or

(b) $\exists\, 1 \leq k \leq l-2$ s.t. $\underline{r_{i_{k-1}} \cdots r_{i_1}(X) \in \{2, 3\}}$ and $(\underline{i_k, i_{k+1}}, i_{k+2}) \in \{ (\underline{2, 3}, 2), (3, 2, 3) \}$.

[$(2, 3, 2)$, $(3, 2, 3)$ are not X_2-reduced X_3-reduced.

We denote by N_X the set of such seq.

(3) $\forall X \in \mathcal{X}$, $\underline{k \notin N_X}$ $\Lambda^X(k) \subset \Lambda_X \cup P_1$

(3)(a) $\underline{X \in \{1, 4\}}$ $\quad \underline{\Lambda^X(k)} \subset P_{i_1}$

(3)(b) $X \in \{2, 3\}$ $\quad i_1 = 1$ $\quad \Lambda^X(k) \subset P_1$

(3)(c) $\underline{X \in \{2, 3\}}$ $\quad \underline{i_1 \in \{2, 3\}}$ $\quad \Lambda^X(k) \subset \Lambda_2$

$X = X_1$ $\qquad \underline{k' = (i_2, i_3 \cdots)}$

i_1	$r_{i_1}(X_1)$	i_2	$\underline{\Lambda(k')}$	$S_{i_1}^{q_{i_1}(X)}$	$\Lambda(k)$
1	X_2	2 3	$\underline{\Lambda_2}$	$\underline{S_1^{X_2}}$	$\underline{P_1}$
2	X_1	1 3	P_1 P_3	$\underline{S_2^{X_1}}$	P_2

$3 \quad X_1 \quad \begin{matrix}1\\2\end{matrix} \quad \begin{matrix}P_1\\P_2\end{matrix} \xrightarrow{S_3^{X_1}} P_3$

$X = X_2$

i_1	$r_{i_1}(X_2)$	i_2	$\Lambda(K')$	$S_{i_1}^{r_{i_1}(X_2)}$	$\Lambda(K)$
1	X_1	2 3	P_2 P_3	$S_1^{X_1}$	P_1
2	X_3	1 3	P_1 $\underline{\Lambda_2}$	$\underline{S_2^{X_3}}$	$\underline{\Lambda_2}$
3	X_2	1 2	P_1 $\underline{\Lambda_2}$	$\underline{S_3^{X_2}}$	$\underline{\Lambda_2}$

$X = X_2 \quad i_1 = 2, \quad r_{i_1}(X_2) = X_3 \quad i_2 = 3, \quad \Lambda(K) \subset \Lambda_2$

$K = (2, \underline{3, 1,} K'''), \quad K' = (3, 1, K'''), \quad K'' = (1, K''')$

$X_2 \xleftarrow{S_2^{X_3}} X_3 \xleftarrow{S_3^{X_3}} X_3 \xleftarrow{S_1^{X_4}} X_4 \leftarrow \cdots$

$X_3 = r_2(X_2)$

$\Lambda(K'') \subset P_1 \qquad \underline{\Lambda(K') =} \{\alpha_3\} \cup S_3^{X_3}(\Lambda(K''))$

$$\begin{aligned}\Lambda(K) &= \{\alpha_2\} \cup S_2^{X_3}(\Lambda(K')) \\ &= \{\alpha_2\} \cup \{S_2^{X_3}(\alpha_3)\} \cup S_2^{X_3} S_3^{X_3}(\underline{\Lambda(K'')}) \\ &\subset \{\alpha_2, \underbrace{S_2^{X_3}(\alpha_3)}_{\Lambda_2}\} \cup \underline{S_2^{X_3} S_3^{X_3}(P_1)} \not\subset \Lambda_2\end{aligned}$$

$\forall\ [a, b, c]^T \in P_1 \qquad S_2^{X_3} S_3^{X_3}(\alpha) = \begin{bmatrix} a \\ 2a - b \\ p_3 a - c \end{bmatrix} \in \Lambda_2$

#